Dynamic Analysis of Multivariate Time Series Using Conditional Wavelet Graphs

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Aims

- Analyze local properties of nonstationary multivariate time series using wavelets (cross correlations)
- Introduce a novel graphical model defined on basis of partial wavelet coherences (PWC)
- Select the graphical model based on observed data: estimate PWC and test for statistical significance

Related Literature

Partial correlation graph for time series

- generalize classical concentration graphs to time series
- restricted to linear dependencies
- accounts for the non-contemporaneous influences (lags)

Brillinger (1981), Brillinger (1996), Dahlhaus (2000), Dahlhaus and Eichler (2003), Eichler (2007), Eckardt (2015) - review study Barigozzi and Brownless (2014)

Graphical Models

A **simple graph** $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of:

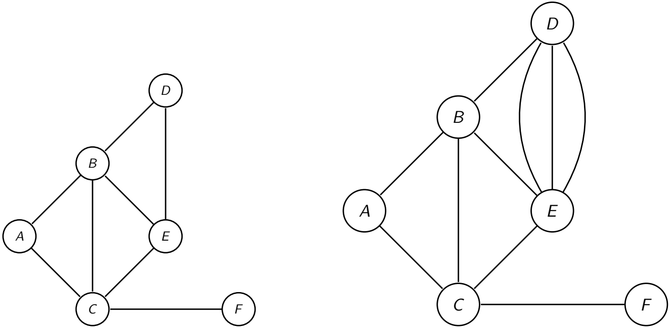
- a set of vertices $\mathcal{V} = \{v_1, \ldots, v_k\} < \infty$
- a set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, $e_{ij} = (v_i, v_j)$
- undirected edges $e_{ij} \in \mathcal{E}(\mathcal{G}) \Leftrightarrow e_{ji} \in \mathcal{E}(\mathcal{G})$
- no graph loops or multiple edges

Usually, $v_i \in \mathcal{V}$ represents a random variable/process

A **multigraph** consists of multiple or parallel edges b/w v_i and v_j.

We derive a loopless undirected multigraph model from which simple graphs can be obtained as subgraphs.

Undirected Graphical Models



(a) MRF (b) Multigraph

Graphical Models for Time Series

k-dimensional **multivariate time series** $X_V(t)$

- $X_V(t) = \{X_i(t)\}_{i \in V},\ t \in \mathbb{Z},\ V = \{1, \ldots, k\}$
- $X_{V \setminus \{i,j\}}(t) = \{X_i(t)\}_{i \in V \setminus \{i,j\}}$.

The **time series graph** of a process X_V

- vertex v_i refers to the X_i component processes of X_V, $\mathcal{V} = k \times \mathbb{Z}$

Linear dependence graphs

- edge e_{ij} is missing if the components X_i and X_j are uncorrelated (given all the other components), i.e. $X_i \perp\!\!\!\perp X_j (| X_{V \setminus \{i,j\}})$ orthogonality (conditional)

Remark: For Gaussian time series - conditional independence.

Partial Correlation Graph for Time Series

Definition: The partial correlation graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ for a stationary process X_V is given by

$$e_{ij} \notin \mathcal{E} \Leftrightarrow X_i \perp\!\!\!\perp X_j \mid X_{V \setminus \{i,j\}}$$
$$\Leftrightarrow cov(\varepsilon_{i|V\setminus\{i,j\}}(t), \varepsilon_{j|V\setminus\{i,j\}}(t+u)), \forall u \in \mathbb{Z}$$

$$\varepsilon_{i|V\setminus\{i,j\}} := X_i(t) - \mu_i^{opt} - \sum_{u=-\infty}^{+\infty} d_i^{opt}(u) X_{V\setminus\{i,j\}}(t-u)$$

$$(\mu_i^{opt}, d_i^{opt}) = \arg\min_{\mu_i, d_i} \mathrm{E}(X_i(t) - \mu_i - \sum_{u=-\infty}^{+\infty} d_i(u) X_{V\setminus\{i,j\}}(t-u))^2$$

Correlation and Partial Correlation Graphs

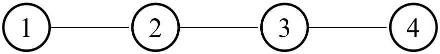
Example: Dahlhaus (2000)

Let $X_1(t) = a_1 X_1(t-1) + \varepsilon_1(t)$,

$$X_j(t) = a_j X_j(t-1) + b_j X_{j-1}(t - t_j) + \varepsilon_j(t)$$

$t_j \in \mathbb{N}$ and $\varepsilon_j(t) \sim \mathrm{N}(0, \sigma)$ iid.

All processes are correlated, i.e. the (simple) correlation graph is complete, while the conditional correlation graph is given bellow



Generalization: If in the partial correlation graph there exist a path between two vertices , then the component processes associated with them are correlated (and vice-versa).

Frequency Domain Formulation

Partial cross-spectrum b/w X_i and X_j at frequency $\omega \in [-\pi, \pi]$

$$f_{ij|V\setminus\{i,j\}}(\omega) = \frac{1}{2\pi} \sum_{t=-\infty}^{+\infty} \left[\sum_{u=-\infty}^{+\infty} \varepsilon_{i|V\setminus\{i,j\}}(t)\varepsilon_{j|V\setminus\{i,j\}}(t+u) \right] e^{-i\omega t}$$

$$= \frac{1}{2\pi} \sum_{u=-\infty}^{+\infty} cov(\varepsilon_{i|V\setminus\{i,j\}}(t), \varepsilon_{j|V\setminus\{i,j\}}(t+u)) e^{-i\omega t}$$

- is the Fourier transform of the cross-correlation function
- is a measure of covariance b/w $\varepsilon_{i|V\setminus\{i,j\}}$ and $\varepsilon_{j|V\setminus\{i,j\}}$

$\rightarrow X_i \perp\!\!\!\perp X_j \mid X_{V\setminus\{i,j\}} \Leftrightarrow f_{ij|V\setminus\{i,j\}}(\omega) = 0, \forall \omega$

Partial Spectral Coherence

Observation: The estimation of residuals $\varepsilon_{i|V\setminus\{i,j\}}(t)$ is computationally intensive.

Alternative: If the spectral matrix $f_V(\omega) = \{f_{ij}(\omega)\}_{i,j\in V}$ is regular and $g(\omega) := f(\omega)^{-1}$ then the partial spectral coherence matrix is $R(\omega) = -diag(g(\omega))^{-1/2} g(\omega) diag(g(\omega))^{-1/2}$, whose elements can be shown to satisfy

$$R_{ij|V\setminus\{i,j\}}(\omega) = \frac{f_{ij|V\setminus\{i,j\}}(\omega)}{\left[f_{ii|V\setminus\{i,j\}}(\omega) f_{jj|V\setminus\{i,j\}}(\omega)\right]^{\frac{1}{2}}}.$$

$\rightarrow X_i \perp\!\!\!\perp X_j \mid X_{V\setminus\{i,j\}} \Leftrightarrow R_{ij|V\setminus\{i,j\}}(\omega) = 0, \forall\omega \Leftrightarrow g(\omega) = f(\omega)^{-1}, \forall\omega$

Localized Partial Correlation Graph

For (possibly) non-ergodic and non-stationary multivariate time series **wavelet**-based methods

- allow time varying analysis of spectral behavior
- characterize dependence in time-frequency domain
- similar to applying linear filters locally $(\mu_{i,t}^{opt}, d_{i,t}^{opt})$ to obtain the errors $\varepsilon_{i|V\setminus\{i,j\}}(t)$
- Similar to local covariance functions, local cross-spectra and local coherence

Remark: If the time series are stationary, their spectral behavior will be constant over time.

Wavelets

- "Mother wavelet" $\psi \in L_2(\mathbb{R})$ s.t.

$$\int_{-\infty}^{\infty} \psi(t)dt = 0 \text{ admissibility condition}$$

$$\int_{-\infty}^{\infty} \psi^2(t)dt = \|\psi\|^2 = 1 \text{ 'unit' energy property.}$$

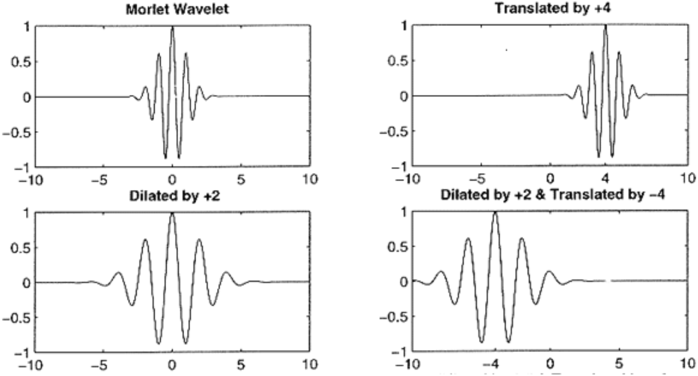
- Families of basis functions $\psi_{\tau,s}(t)$

$$\psi_{\tau,s}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-\tau}{s}\right), \ s \in \mathbb{R}^+, \tau \in \mathbb{R} \qquad (1)$$

  τ location and s scale (pseudo-frequency); $\|\psi_{\tau,s}\| = 1$

Note: We will consider complex wavelets further on.

Example: Morlet Wavelet



Morlet wavelet under translation and dilation

Wavelet Transform

Wavelet coefficients w.r.t. X_i

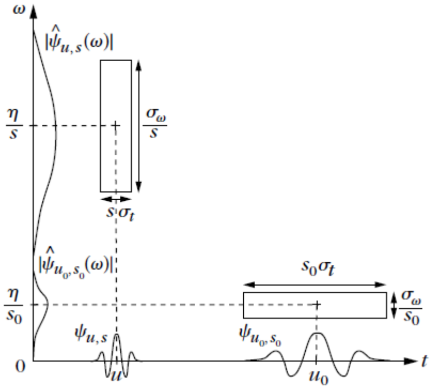
$$W_i(\tau, s) = \langle X_i, \psi_{\tau,s} \rangle = \frac{1}{\sqrt{s}} \sum_{-\infty}^{+\infty} X_i(t)\overline{\psi_{\tau,s}(t)}$$

$\overline{(\cdot)}$ stands for the complex conjugate. Additionally, a frequency domain representation of $W_i(\tau, s)$ follows as

$$W_i(\omega) = \frac{\sqrt{|s|}}{2\pi} \sum_{t=-\infty}^{\infty} X_i(t)\overline{f_{\psi_{s,\tau}}(st)}e^{i\omega t},$$

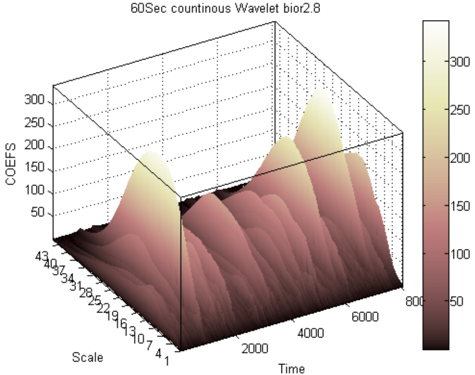
where $f_{\psi_{s,\tau}}$ is the Fourier transform of the wavelet function $\psi_{\tau,s}(t)$.

'Adaptive' Window



Heisenberg time-frequency boxes of two wavelet basis

Scaleogram



Arousal-valence scale for the EEG signal, Sorkhabi (2014)

Parseval's Relation: Extension to Wavelets

Recall: The inner product of two time series equals the inner product of their Fourier transform.

- $X_i(t)$ can be recovered from the wavelet transform

$$X_i(t) = \frac{1}{C_\psi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{s^2} W_i(\tau, s) \psi_{\tau,s}(t) d\tau ds$$

- For two processes $X_i(t)$ and $X_j(t)$, the energy in the time domain is preserved in the time-frequency domain

$$\langle X_i X_j \rangle = \frac{1}{C_\psi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{s^2} |W_i(\tau, s) \overline{W_i(\tau, s)}| d\tau ds,$$

  for a finite constant C_ψ satisfying

$$C_\psi = \int_{-\infty}^{\infty} \frac{|\psi(\omega)|^2}{|\omega|} d\omega < \infty.$$

Partial Cross Wavelet

- Cross-wavelet coefficients - can be interpreted as a localized measure of correlation between two time series

$$W_{ij}(\tau, s) = W_i(\tau, s)\overline{W_j}(\tau, s)$$

- Partial cross-wavelet

$$W_{ij|V\setminus\{i,j\}}(\tau, s) = W_{ij}(\tau, s) - W_{iV\setminus\{i,j\}}(\tau, s) W_{V\setminus\{i,j\}V\setminus\{i,j\}}(\tau, s)^{-1} W_{jV\setminus\{i,j\}}(\tau, s)$$

It extends a result for partial cross-spectrum (Brillinger, 1981) and involves inversion of $(k-2) \times (k-2)$ dimensional matrix; alternatively solve via recursion formula.

Partial Wavelet Coherence

- Partial wavelet coherence

$$R_{ij|V\setminus\{i,j\}}(\tau, s) = \frac{|W_{ij|V\setminus\{i,j\}}(\tau, s)|}{|W_{ii|V\setminus\{i,j\}}(\tau, s) W_{jj|V\setminus\{i,j\}}(\tau, s)|^{\frac{1}{2}}}$$

$0 \le |R_{ij|V\setminus\{i,j\}}(\tau, s)|^2 \le 1$, interpreted as a localized correlation in the time-frequency domain

Remark. $X_i \perp\!\!\!\perp X_j \mid X_{V\setminus\{i,j\}} \Leftrightarrow R_{ij|V\setminus\{i,j\}}(\tau, s) = 0, \forall s, \tau \Leftrightarrow |W_{ij|V\setminus\{i,j\}}(\tau, s)| = 0, \forall s, \tau$

Wavelet Dependence Graph

For $X_V(t)$ a multivariate stochastic process evolving in discrete time a *wavelet dependence graph* is an undirected multigraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ in which any $v_i \in \mathcal{V}(\mathcal{G})$ encodes the i-th component $X_i(t)$ of $X_V(t)$ s.t.

$$X_{i,s} \perp\!\!\!\perp X_{j,s} \mid X_{V\setminus\{i,j\},s} \Leftrightarrow e_{ij,s} \notin \mathcal{E}_s(\mathcal{G})$$
$$\Leftrightarrow R_{ij|V\setminus\{i,j\}}(\tau, s) = 0, \forall \tau$$

at fixed scale s, where $\mathcal{E}_s(\mathcal{G})$ is a scale-specific subset and it holds that $\mathcal{E}(\mathcal{G}) = \cup \mathcal{E}_s(\mathcal{G})$.

Remark: A partial correlation (wavelet) graph can be obtained from the multigraph by replacing any multiedge by at most one edge.

Outlook

Graph estimation

- noisy observation: shrinkage/smoothing of the wavelet coefficients, LASSO
- distributional assumptions for testing: Gaussian errors, Monte-Carlo methods

Extensions

- Directed graphs - Granger causality
- Dynamic graphs
- Simulation, real data

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